Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **58**, 1–17; **http://www.math.u-szeged.hu/ejqtde/**

Removability of singularity for nonlinear elliptic equations with $p(x)$-growth*

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Abstract

Using Moser's iteration method, we investigate the problem of removable isolated singularities for elliptic equations with $p(x)$-type nonstandard growth. We give a sufficient condition for removability of singularity for the equations in the framework of variable exponent Sobolev spaces.

Keywords: variable exponent space; isolated singularity; removable singularity.
2010 Mathematics Subject Classification: 35B60; 35J60.

1 Introduction

In recent years, the research of elliptic equations with variable exponent growth conditions has been an interesting topic. These problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator $-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is inhomogeneous, and these problems have many important applications, see [1, 2, 3]. Since Kováčik and Rákosník first studied the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [4], many results have been obtained concerning these kinds of variable exponent spaces, see examples in $[5-12]$.

In this paper, we study solutions to nonlinear elliptic equations with nonstandard growth in the divergence form

$$-\text{div}A(x,u,\nabla u)+g(x,u)=0. \tag{1.1}$$

in a punctured domain $\Omega\setminus\{0\}$, where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary.

Throughout the paper we suppose that the functions $A(\cdot,\xi,\eta):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $g(\cdot,\xi):\Omega\times\mathbb{R}\to\mathbb{R}^N$ are measurable for all $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^N$, and $A(x,\cdot,\cdot)$, $g(x,\cdot)$ are continuous for almost all $x\in\Omega$. We also assume that the following structure conditions

$$A(x,\xi,\eta)\eta\geq\mu_1|\eta|^{p(x)}, \tag{1.2}$$

*This work was supported by the National Natural Science Foundation of China (Grant No. 11371110).
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$$|A(x,\xi,\eta)| \le \mu_2 |\eta|^{p(x)-1}, \tag{1.3}$$

$$A(x,\xi,-\eta) = -A(x,\xi,\eta) \tag{1.4}$$

$$|x|^{-\alpha}|\xi|^{q(x)} \le g(x,\xi)\mathrm{sgn}\xi \le C|x|^{-\alpha}|\xi|^{q(x)} \tag{1.5}$$

are fulfilled for almost all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^N$, where $\mu_1, \mu_2 > 0, \alpha < N, C > 1$ are constants, $p, q \in C(\overline{\Omega})$, $1 < p^- \le p(x) \le p^+ < N$, and $q(x) \gg p(x) - 1$.

Here we denote

$$p^- = \inf_{x\in\overline{\Omega}} p(x), \quad p^+ = \sup_{x\in\overline{\Omega}} p(x),$$

and denote by $q(x) \gg p(x) - 1$ the fact that $\inf_{x\in\overline{\Omega}}(q(x) - p(x) + 1) > 0$.

For the Laplace's equation, a set of capacity zero constitutes a removable singularity for a bounded harmonic function, while, a single point x_0 is removable if the solution is $o(\log|x-x_0|)$ or $o(|x-x_0|^{2-N})$.

Serrin [13] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of $g(x,u) \equiv 0$, it is shown that at an isolated singularity a positive solution has precisely the order of growth $|x-x_0|^{\frac{p-N}{p-1}}$ if $1 < p < N$, or $\log\frac{1}{|x-x_0|}$ if $p = N$.

Brezis and Veron [14] studied the equation of form (1.1) with a Laplace operator in the principal part. They proved the removability of isolated singularities for solutions under condition $g(x,\xi)\mathrm{sgn}\xi \ge |\xi|^q$ and $q \ge \frac{N}{N-2}, N \ge 3$.

For the equation of the form:

$$-\mathrm{div}A(x,u,\nabla u) + a_0(x,u,\nabla u) = 0$$

Serrin [13, 15] considered the conditions of removability of an isolated singular point x_0, the condition has the form

$$u(x) = o\left(|x-x_0|^{\frac{p-N}{p-1}+\tau}\right), \quad 1 < p < N,$$

with positive number τ. Nicolosi et al. [16] obtained a precise condition for the removability of singularities, it has the form

$$u(x) = o\left(|x-x_0|^{\frac{p-N}{p-1}}\right), \quad 1 < p < N.$$

For equations with weighted functions v, w, Mamedov and Harman [17] proved that an isolated singular point x_0 is removable for solutions of equation (1.1) if the condition of weighted functions

$$v(B(x_0,\varepsilon))\left(\frac{w(B(x_0,\varepsilon))}{\varepsilon^p v(B(x_0,\varepsilon))}\right)^{\frac{q}{q-p+1}} = o(1), \qquad \varepsilon \to 0,$$

and $p > 1$, $q > p-1$ are fulfilled. For the removability of singularities for solutions of elliptic equations with absorption term (see [18, 19]).

Recently, there have been a few papers on the study of the removability of singularities for the equations with nonstandard growth. Lukkari [20] investigated the removability of a compact set for the equation $-\text{div}\left(|Du|^{p(x)-2}Du\right) = 0$. For the anisotropic elliptic equation, the removability of a compact set was proved by Cianci [21]. Cataldo and Cianci [22] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of $g(x,u) = |u|^{q-2}u$.

In this paper, following Moser's method [23], we establish the condition

$$1 < \frac{(p(x)-\alpha)q(x)}{q(x)-p(x)+1} + \alpha \ll N \quad \text{a.e. on } \overline{\Omega} \tag{1.6}$$

to ensure the removability of singularities.

2 Preliminaries

We first recall some facts on spaces $L^{p(x)}$ and $W^{k,p(x)}$. For the details see [4, 8].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$, we denote

$$\rho_{p(x)}(u) = \int_{\Omega\setminus\Omega_\infty} |u|^{p(x)}\,dx + \sup_{x\in\Omega_\infty} |u(x)|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu) < \infty$, for some $t > 0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^{p(x)}} = \inf\{\lambda > 0 : \rho_{p(x)}\left(\frac{u}{\lambda}\right) \le 1\}.$$

For any $p \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 = \{x \in \Omega : p(x) = 1\}, \\ 1, & x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1}, & x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty). \end{cases}$$

Theorem 2.1 *Let $p \in \mathbf{P}(\Omega)$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int_\Omega |uv|\,dx \le 2\|u\|_{L^{p(x)}}\|v\|_{L^{p'(x)}}.$$

Theorem 2.2 *Let $p \in \mathbf{P}(\Omega)$ with $p^+ < \infty$. For any $u \in L^{p(x)}(\Omega)$, we have*

(1) if $\|u\|_{L^{p(x)}} \ge 1$, then $\|u\|^{p^-}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^+}_{L^{p(x)}}$;

(2) if $\|u\|_{L^{p(x)}} < 1$*, then* $\|u\|^{p^+}_{L^{p(x)}} \leq \int_\Omega |u|^{p(x)}\, dx \leq \|u\|^{p^-}_{L^{p(x)}}$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}} + \|\nabla u\|_{L^{p(x)}}.$$

We say that the function $u(x)$ belongs to the space $W^{1,p(x)}_{loc}(\Omega)$ if $u(x)$ belongs to $W^{1,p(x)}(G)$ in any subdomain G, $\overline{G} \subset \Omega$.

Theorem 2.3 *For any* $u \in W^{1,p(x)}(\Omega)$*, we have*

(1) if $\|u\|_{W^{1,p(x)}} \geq 1$*, then* $\|u\|^{p^-}_{W^{1,p(x)}} \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \leq \|u\|^{p^+}_{W^{1,p(x)}}$;

(2) if $\|u\|_{W^{1,p(x)}} < 1$*, then* $\|u\|^{p^+}_{W^{1,p(x)}} \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \leq \|u\|^{p^-}_{W^{1,p(x)}}$.

From Zhikov [5, 6], we know smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. To study the Lavrentiev phenomenon, he considered the following log-Hölder continuous condition

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)} \tag{2.1}$$

for all $x, y \in \overline{\Omega}$ such that $|x-y| \leq \frac{1}{2}$. If the log-Hölder continuous condition holds, then smooth functions are dense in $W^{1,p(x)}(\Omega)$ and we can define the Sobolev spaces with zero boundary values $W^{1,p(x)}_0(\Omega)$, as the closure of $C^\infty_0(\Omega)$ with the norm of $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.

Theorem 2.4 *If* $u \in W^{1,p}_0(B_R(a))$*,* $1 \leq p < N$*, then for any* $1 \leq q \leq p^*$*, the inequality*

$$\left(\int_{B_R(a)} |u|^q dx\right)^{\frac{1}{q}} \leq C(N,p) R^{1+\frac{N}{q}-\frac{N}{p}} \left(\int_{B_R(0)} |Du|^p dx\right)^{\frac{1}{p}} \tag{2.2}$$

is valid, where $B_R(a)$ *is the ball of radius* R *with centre* a.

We define $p^+_\delta = \sup\limits_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} p(y)$, $p^-_\delta = \inf\limits_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} p(y)$, $q^+_\delta = \sup\limits_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} q(y)$, $q^-_\delta = \inf\limits_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} q(y)$, where $\delta > 0$ is a constant.

Lemma 2.1 *Since* $q(x) \gg p(x) - 1$*, then the set* $S = \{\delta : p^+_\delta - 1 < q^-_\delta\}$ *is nonempty, bounded above and* $\delta_0 = \sup\{\delta : p^+_\delta - 1 < q^-_\delta\} < +\infty$.

Proof. As $q(x)$, $p(x)$ are continuous on $\overline{\Omega}$, for $\varepsilon_1 \in (0,1)$ and $0 \in \Omega$, there exists $\delta > 0$ such that $|q(0) - q(y)| < \varepsilon_1$ and $|p(0) - p(y)| < \varepsilon_1$ whenever $|y| < \delta$. For any $y \in B_\delta(0) \cap \overline{\Omega}$, we have

$$p(y) - 1 < p(0) - 1 + \varepsilon_1,$$

and

$$q(y) > q(0) - \varepsilon_1.$$

As $q(x) \gg p(x) - 1$, take $\varepsilon_1 = \frac{1}{4} \inf\limits_{x \in \overline{\Omega}} (q(x) - p(x) + 1)$,

$$q(0) - \varepsilon_1 - (p(0) - 1 + \varepsilon_1) \geq \frac{1}{2} \inf_{x \in \overline{\Omega}} (q(0) - p(0) + 1) > 0,$$

then

$$p(y) - 1 < p(0) - 1 + \varepsilon_1 < q(0) - \varepsilon_1 < q(y),$$

and further

$$p_\delta^+ - 1 = \sup_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} (p(y) - 1) < q_\delta^- = \inf_{y \in \overline{B_\delta(0)} \cap \overline{\Omega}} q(y).$$

So the set $S = \{\delta : p_\delta^+ - 1 < q_\delta^-\}$ is nonempty. From the definition of the $q(x) \gg p(x) - 1$, we know the set S is bounded above. By the Continuum Property, it has a smallest upper bound δ_0. This smallest upper bound δ_0 is called the supremum of the set S. We write $\delta_0 = \sup S = \sup\{\delta : p_\delta^+ - 1 < q_\delta^-\}$.

Consider a solution $u(x)$ of equation (1.1) with an isolated singularity. Assume that $0 \in \Omega$ and zero is a singular point of the solution $u(x)$. We say that $u(x)$ is a solution of equation (1.1) in $\Omega \setminus \{0\}$ if $u \in W^{1,p(x)}(\Omega \setminus \{0\})$ and for any test function $\varphi \in W_0^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\})$ in $\Omega \setminus \{0\}$, the following equality is true:

$$\int_\Omega (A(x, u, \nabla u)\nabla\varphi + g(x, u)\varphi)\, dx = 0. \tag{2.3}$$

We say that the solution $u(x)$ of equation (1.1) has a removable singularity at the point 0 if the function $u(x)$ is a solution in $\Omega \setminus \{0\}$ and $u \in W^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\})$ implies that it belongs to the space $W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and satisfies (2.3) for any test function $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

3 Proof of theorems

In this section we state and prove the following theorems.
In the sequel by C we denote a constant, the value of which may vary from line to line.

Theorem 3.1 *Let $u \in W^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\})$ be a solution of equation (1.1) in $\Omega \setminus \{0\}$. Assume that conditions (1.2) – (1.5), (2.1) are satisfied. Then for any $0 < |x| \le R < \min\{dist(0, \partial\Omega), \delta_0, 1\}$, the estimate*

$$|u(x)| \le C|x|^{-Q}, \tag{3.1}$$

holds almost everywhere, where $Q = Q(N, \alpha, p_R^-, p_R^+, q_R^-)$ and $C = C(N, \mu_1, \mu_2, p_R^-, p_R^+, q_R^-, q_R^+, R)$.

Proof. For $\rho < R$ we define a smooth cut-off function $\varphi_1(x)$ satisfying conditions: $\varphi_1(x) = 1$ for $\frac{\rho}{2} < |x| < \frac{3\rho}{4}$, $\varphi_1(x) = 0$ outside the set for $\frac{\rho}{4} \le |x| \le \rho$, $|\nabla\varphi_1(x)| \le \frac{C}{\rho}$ and $0 \le \varphi_1(x) \le 1$.

Take the test function

$$\psi(x) = (1 + |u(x)|)^m u(x)\varphi_1(x)^{n+p_R^+} \in W_0^{1,p(x)}(B_R(0)\backslash\{0\}),$$

$m, n \ge 0$ are nonnegative numbers to be determined later, and then

$$\begin{aligned}\nabla\psi(x) &= m(1 + |u(x)|)^{m-1}\nabla u(x)|u(x)|\varphi_1(x)^{n+p_R^+} + (1 + |u(x)|)^m \nabla u(x)\varphi_1(x)^{n+p_R^+} \\ &\quad + (1 + |u(x)|)^m u(x)(n + p_R^+)\varphi_1(x)^{n+p_R^+-1}\nabla\varphi_1(x).\end{aligned}$$

We substitute the test function $\psi(x)$ into the integral identity (2.3), we obtain

$$\begin{aligned}&\int_{B_R(0)} mA(x, u, \nabla u)(1 + |u(x)|)^{m-1}\nabla u(x)|u(x)|\varphi_1(x)^{n+p_R^+}dx \\ &+ \int_{B_R(0)} A(x, u, \nabla u)(1 + |u(x)|)^m \nabla u(x)\varphi_1(x)^{n+p_R^+}dx \\ &+ \int_{B_R(0)} g(x, u)(1 + |u(x)|)^m u(x)\varphi_1(x)^{n+p_R^+}dx \\ &+ \int_{B_R(0)} A(x, u, \nabla u)(1 + |u(x)|)^m u(x)(n + p_R^+)\varphi_1(x)^{n+p_R^+-1}\nabla\varphi_1(x)dx = 0.\end{aligned}$$

By virtue of the conditions (1.2) – (1.5),

$$\begin{aligned}&\int_{B_R(0)} \mu_1 m|\nabla u(x)|^{p(x)}(1 + |u(x)|)^{m-1}|u(x)|\varphi_1(x)^{n+p_R^+}dx \\ &+ \int_{B_R(0)} \mu_1|\nabla u(x)|^{p(x)}(1 + |u(x)|)^m\varphi_1(x)^{n+p_R^+}dx \\ &+ \int_{B_R(0)} |x|^{-\alpha}|u(x)|^{q(x)+1}(1 + |u(x)|)^m\varphi_1(x)^{n+p_R^+}dx \\ &\le \int_{B_R(0)} \mu_2(n + p_R^+)|\nabla u(x)|^{p(x)-1}(1 + |u(x)|)^{m+1}\varphi_1(x)^{n+p_R^+-1}|\nabla\varphi_1(x)|dx,\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
&\int_{B_R(0)} \mu_1 |\nabla u(x)|^{p(x)} \left(1+|u(x)|\right)^m \varphi_1(x)^{n+p_R^+} dx + \int_{B_R(0)} |x|^{-\alpha} |u(x)|^{q(x)+m+1} \varphi_1(x)^{n+p_R^+} dx \\
&\le \mu_2 \int_{B_R(0)} \left(1+|u(x)|\right)^m \varphi_1(x)^{n+p_R^+} \left[|\nabla u(x)|^{p(x)-1}\right] \left[(n+p_R^+)(1+|u(x)|)\varphi_1(x)^{-1}|\nabla\varphi_1(x)|\right] dx \\
&\le \mu_2 C(\varepsilon_2) \int_{B_R(0)} (n+p_R^+)^{p(x)} (1+|u(x)|)^{p(x)+m} \varphi_1(x)^{n+p_R^+-p(x)} |\nabla \varphi_1(x)|^{p(x)} dx \\
&\quad + \mu_2 \varepsilon_2 \int_{B_R(0)} (1+|u(x)|)^m \varphi_1(x)^{n+p_R^+} |\nabla u(x)|^{p(x)} dx
\end{aligned}$$

Take $\varepsilon_2 = \frac{\mu_1}{2\mu_2}$, we have

$$\begin{aligned}
&\frac{\mu_1}{2} \int_{B_R(0)} |\nabla u(x)|^{p(x)} (1+|u(x)|)^m \varphi_1(x)^{n+p_R^+} dx + \int_{B_R(0)} |x|^{-\alpha} |u(x)|^{q(x)+m+1} \varphi_1(x)^{n+p_R^+} dx \\
&\le C(\mu_1,\mu_2) \int_{B_R(0)} (n+p_R^+)^{p(x)} \frac{1}{\rho^{p(x)}} \left(1+|u(x)|\right)^{p(x)+m} \varphi_1(x)^{n+p_R^+-p(x)} dx.
\end{aligned} \tag{3.2}$$

Denote $p_R^{-*} = \frac{Np_R^-}{N-p_R^-} = kp_R^-$. Since $u(x) \in W^{1,p(x)}(B_R(0)\backslash\{0\})$, then $u(x) \in W^{1,p_R^-}(B_R(0)\backslash\{0\})$ and $\phi(x) = \left[(1+|u(x)|)^{t+p_R^+} \varphi_1(x)^{s+p_R^+}\right]^{\frac{1}{kp_R^-}} \in W_0^{1,p_R^-}(B_R(0))$, where $t+p_R^+ > kp_R^-$, $s+p_R^+ > kp_R^+$. As $1 < p_R^- < N$, applying (2.2) to the function $\phi(x)$, we have

$$\begin{aligned}
&\int_{B_R(0)} (1+|u(x)|)^{t+p_R^+} \varphi_1(x)^{s+p_R^+} dx \\
&\le C(N,p_R^-) \left(\int_{B_R(0)} |\nabla \phi(x)|^{p_R^-} dx\right)^k \\
&= C(N,p_R^-) \left\{ \int_{B_R(0)} \left[\left(\frac{t+p_R^+}{kp_R^-}\right)^{p_R^-} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-} |\nabla u(x)|^{p_R^-} \varphi_1^{\frac{s+p_R^+}{k}} \right.\right. \\
&\quad \left.\left. + \left(\frac{s+p_R^+}{kp_R^-}\right)^{p_R^-} (1+|u(x)|)^{\frac{t+p_R^+}{k}} \varphi_1^{\frac{s+p_R^+}{k}-p_R^-} |\nabla\varphi_1|^{p_R^-} \right] dx \right\}^k \\
&\le C(N,p_R^-) \left(\frac{t+s+p_R^+}{kp_R^-}\right)^{kp_R^-} \left\{ \int_{B_R(0)} \left[(1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-} |\nabla u(x)|^{p_R^-} \varphi_1^{\frac{s+p_R^+}{k}} \right.\right. \\
&\quad \left.\left. + \left(\frac{1}{\rho}\right)^{p_R^-} (1+|u(x)|)^{\frac{t+p_R^+}{k}} \varphi_1^{\frac{s+p_R^+}{k}-p_R^-} \right] dx \right\}^k.
\end{aligned} \tag{3.3}$$

Taking $m = \frac{t+p_R^+}{k} - p_R^-$, $n + p_R^+ = \frac{s+p_R^+}{k}$ in (3.2) and using Young's inequality, we have

$$
\begin{aligned}
&\int_{B_R(0)} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-} |\nabla u(x)|^{p_R^-} \varphi_1^{\frac{s+p_R^+}{k}} dx \\
&\leq \int_{B_R(0)} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-} |\nabla u(x)|^{p(x)} \varphi_1^{\frac{s+p_R^+}{k}} dx + \int_{B_R(0)} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-} \varphi_1^{\frac{s+p_R^+}{k}} dx \qquad (3.4)\\
&\leq C(\mu_1,\mu_2)\left(s+p_R^+\right)^{p_R^+} \frac{1}{\rho^{p_R^+}} \int_{B_R(0)} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-+p(x)} \varphi_1^{\frac{s+p_R^+}{k}-p(x)} dx.
\end{aligned}
$$

From (3.3) and (3.4) we get

$$
\begin{aligned}
&\int_{B_R(0)} (1+|u(x)|)^{t+p_R^+} \varphi_1(x)^{s+p_R^+} dx \\
&\leq C(s+p_R^+)^{kp_R^+}\left(t+s+p_R^+\right)^{kp_R^-} \frac{1}{\rho^{kp_R^+}} \left[\int_{B_R(0)} (1+|u(x)|)^{\frac{t+p_R^+}{k}-p_R^-+p_R^+} \varphi_1^{\frac{s+p_R^+}{k}-p_R^+} dx\right]^k, \qquad (3.5)
\end{aligned}
$$

where $C = C(N, \mu_1, \mu_2, p_R^+, p_R^-)$.

Denote

$$
\begin{aligned}
I_i &= \int_{B_R(0)} (1+|u(x)|)^{t_i+p_R^+} \varphi_1(x)^{s_i+p_R^+} dx, \\
t_i &= (q_R^- + kp_R^-)k^i - p_R^+ + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}, \\
s_i &= \left(s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)k^i - p_R^+ - \frac{Np_R^+}{p_R^-},
\end{aligned}
$$

where

$$
s_0 = \frac{p_R^+\left(q_R^+ + kp_R^- + \frac{(p_R^+-p_R^-)N}{p_R^-+1}\right)}{q_R^- - p_R^+ + 1} - p_R^+ + 1.
$$

From (3.5), we get

$$
I_i \leq C(N,\mu_1,\mu_2,p_R^+,p_R^-)\left(t_i+s_i+p_R^+\right)^{2kp_R^+} \frac{1}{\rho^{kp_R^+}} I_{i-1}^k. \qquad (3.6)
$$

Since

$$
\begin{aligned}
t_i + s_i + p_R^+ &\leq \left(q_R^- + kp_R^-\right)k^i + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-} + \left(s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)k^i - \frac{Np_R^+}{p_R^-} \\
&\leq \left(q_R^- + kp_R^- + s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)k^i,
\end{aligned}
$$

iterate (3.6), then we have

$$I_i \le C\left(q_R^- + kp_R^- + s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)^{2kp_R^+} k^{2kip_R^+}\frac{1}{\rho^{kp_R^+}} I_{i-1}^k$$
$$\le C\left(q_R^- + kp_R^- + s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)^{2\sum\limits_{j=1}^{i} k^j p_R^+} k^{2\sum\limits_{j=1}^{i}(i+1-j)k^j p_R^+}\left(\frac{1}{\rho}\right)^{\sum\limits_{j=1}^{i} k^j p_R^+} I_0^{k^i},$$

then

$$\begin{aligned}&\left[\int_{B_R(0)} (1+|u(x)|)^{\left(q_R^- + kp_R^-\right)k^i + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}} \varphi_1(x)^{s_i + p_R^+} dx\right]^{\frac{1}{k^i}} \\ &\le C\left(q_R^- + kp_R^+ + s_0 + p_R^+ + \frac{Np_R^+}{p_R^-}\right)^{2\sum\limits_{j=1}^{i} k^{j-i} p_R^+} k^{2\sum\limits_{j=1}^{i}(i+1-j)k^{j-i} p_R^+}\left(\frac{1}{\rho}\right)^{\sum\limits_{j=1}^{i} k^{j-i} p_R^+} I_0,\end{aligned} \tag{3.7}$$

where $C = C(N, \mu_1, \mu_2, p_R^+, p_R^-)$.

Since

$$\begin{aligned}\left[\int_{B_R(0)} |u(x)|^{q_R^- k^i} \varphi_1(x)^{s_i+p_R^+} dx\right]^{\frac{1}{k^i}} &\le \left[\int_{B_R(0)} (1+|u(x)|)^{q_R^- k^i} \varphi_1(x)^{s_i+p_R^+} dx\right]^{\frac{1}{k^i}} \\ &\le \left[\int_{B_R(0)} (1+|u(x)|)^{\left(q_R^- + kp_R^-\right)k^i + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}} \varphi_1(x)^{s_i+p_R^+} dx\right]^{\frac{1}{k^i}},\end{aligned} \tag{3.8}$$

combining (3.7) and (3.8), and passing to the limit as $i \to \infty$, we obtain

$$\begin{aligned}||u(x)||^{q_R^-}_{L^\infty\left(\frac{\rho}{2}<|x|<\frac{3\rho}{4}\right)} &\le \| \, 1+|u(x)| \, \|^{q_R^-}_{L^\infty\left(\frac{\rho}{2}<|x|<\frac{3\rho}{4}\right)} \\ &\le C\left(\frac{1}{\rho}\right)^{\frac{kp_R^+}{k-1}} \left[\int_{B_R(0)} (1+|u(x)|)^{q_R^- + kp_R^- + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}} \varphi_1(x)^{s_0+p_R^+} dx\right],\end{aligned} \tag{3.9}$$

where $C = C(N, \mu_1, \mu_2, p_R^+, p_R^-)$.

Taking $m = kp_R^- + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}$, $n = s_0$ in (3.2), we have

$$\begin{aligned}&\int_{B_R(0)} |x|^{-\alpha} |u(x)|^{q(x)+kp_R^- + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}+1} \varphi_1(x)^{s_0+p_R^+} dx \\ &\le C(N, \mu_1, \mu_2, p_R^+, p_R^-) \int_{B_R(0)} \frac{1}{\rho^{p(x)}} (1+|u(x)|)^{p(x)+kp_R^- + \frac{\left(p_R^+ - p_R^-\right)N}{p_R^-}} \varphi_1(x)^{s_0+p_R^+-p(x)} dx,\end{aligned} \tag{3.10}$$

and further by (3.10), we get

$$\int_{B_R(0)} (1+|u(x)|)^{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1} \varphi_1(x)^{s_0+p_R^+} dx$$

$$\leq C(N,p_R^+,p_R^-,q_R^+)\int_{B_R(0)} \left(1+|u(x)|^{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1}\right)\varphi_1^{s_0+p_R^+} dx$$

$$\leq C + C\int_{B_R(0)} \rho^{\alpha-p_R^+} (1+|u(x)|)^{p(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}} \varphi_1^{s_0+p_R^+-p(x)} dx$$

$$\leq C + C\varepsilon_3\int_{B_R(0)} (1+|u(x)|)^{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1} \varphi_1(x)^{s_0+p_R^+} dx +$$

$$C(\varepsilon_3)\int_{B_R(0)} \rho^{(\alpha-p_R^+)\frac{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1}{q(x)-p(x)+1}} \varphi_1^{s_0+p_R^+-\frac{p(x)\left(q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-+1}\right)}{q(x)-p(x)+1}} dx.$$

Take $\varepsilon_3=\frac{1}{2C}$, we have

$$\int_{B_R(0)} (1+|u(x)|)^{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1} \varphi_1(x)^{s_0+p_R^+} dx$$

$$\leq C\left(1+\int_{B_R(0)} \rho^{(\alpha-p_R^+)\frac{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1}{q(x)-p(x)+1}} dx\right),$$

where $C=C(N,\mu_1,\mu_2,p_R^+,p_R^-,q_R^+,R)$.

From (3.9), we have

$$||u(x)||^{q_R^-}_{L^\infty\left(\frac{\rho}{2}<|x|<\frac{3\rho}{4}\right)} \leq C\left(\rho^{-\frac{kp_R^+}{k-1}}+\rho^{-\frac{kp_R^+}{k-1}}\int_{B_R(0)} \rho^{(\alpha-p_R^+)\frac{q(x)+kp_R^-+\frac{(p_R^+-p_R^-)N}{p_R^-}+1}{q(x)-p(x)+1}} dx\right). \tag{3.11}$$

If $p_R^+\leq\alpha<N$, we have

$$||u(x)||^{q_R^-}_{L^\infty\left(\frac{\rho}{2}<|x|<\frac{3\rho}{4}\right)} \leq C\rho^{-\frac{kp_R^+}{k-1}},$$

and

$$|u(x)|\leq C|x|^{-\frac{kp_R^+}{(k-1)q_R^-}}, \quad \text{a.e.}$$

where $C=C\left(N,\mu_1,\mu_2,p_R^+,p_R^-,q_R^+,q_R^-,R\right)$.

If $\alpha < p_R^+$, we have

$$||u(x)||^{q_R^-}_{L^\infty\left(\frac{\varrho}{2}<|x|<\frac{3\varrho}{4}\right)} \le C\rho^{-\frac{\left(p_R^+-\alpha\right)\left(q_R^++kp_R^-+\frac{\left(p_R^+-p_R^-\right)N}{p_R^-}+1\right)}{q_R^--p_R^++1}-\frac{kp_R^+}{k-1}},$$

and

$$|u(x)| \le C\,|x|^{-\left\{\frac{\left(p_R^+-\alpha\right)\left(q_R^++kp_R^-+\frac{\left(p_R^+-p_R^-\right)N}{p_R^-}+1\right)}{\left(q_R^--p_R^++1\right)q_R^-}+\frac{kp_R^+}{(k-1)q_R^-}\right\}}, \quad \text{a.e.}$$

where $C = C\left(N, \mu_1, \mu_2, p_R^+, p_R^-, q_R^+, q_R^-, R\right)$.

The following is the main theorem in this paper.

Theorem 3.2 *Let conditions* $(1.2) - (1.6)$, (2.1) *be fulfilled. If u is a solution of equation* (1.1) *in $\Omega \setminus \{0\}$, then the singularity of $u(x)$ at the point 0 is removable.*

Proof. For $0 < r < R < \min\{\mathrm{dist}(0, \partial\Omega), \delta_0, 1\}$, we denote $m(r) = \sup\{|u(x)| : r \le |x| \le R\}$. For sufficiently small $r \le \min\left\{\frac{1}{e^2}, R^2\right\}$, we define the function $\psi_r(x)$ as follows:

$$\begin{aligned}
&\psi_r(x) \equiv 0 \qquad \text{for} \quad |x| < r,\\
&\psi_r(x) \equiv 1 \qquad \text{for} \quad |x| > \sqrt{r},\\
&\psi_r(x) = \frac{2}{\ln\frac{1}{r}} \ln\frac{|x|}{r} \qquad \text{for} \quad r \le |x| \le \sqrt{r}.
\end{aligned}$$

We take the following test function

$$\varphi(x) = \psi_r^\gamma(x)\left[\ln\frac{u}{m(\varrho)}\right]_+, \tag{3.12}$$

for any $x \in \Omega_\varrho$, where $0 < \varrho < R$, $\Omega_\varrho = \{x \in B_R(0) : u(x) > m(\varrho)\}$, $\gamma = \sup\limits_{x\in\overline{\Omega}} \frac{p(x)q(x)}{q(x)-p(x)+1}$ is a constant and $\varphi(x) \equiv 0$ for $x \notin \Omega_\varrho$.

For some $0 < \varrho < R$, let the domain Ω_ϱ be nonempty. Since $\varphi(x) \in W_0^{1,p(x)}(\Omega\backslash\{0\}) \cap L^\infty(\Omega\backslash\{0\})$, testing the equality (2.3) by φ, we have

$$\begin{aligned}
&\int_{\Omega_\varrho} A(x, u, \nabla u)\nabla u\frac{\psi_r^\gamma}{u} + g(x, u)\psi_r^\gamma(x)\ln\frac{u}{m(\varrho)}dx\\
&+\int_{\Omega_\varrho} A(x, u, \nabla u)\gamma\psi_r^{\gamma-1}(x)\nabla\psi_r\ln\frac{u}{m(\varrho)}dx = 0.
\end{aligned}$$

By virtue of the conditions $(1.2)-(1.4)$, we have

$$\begin{aligned}&\int_{\Omega_\varrho}\mu_1\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx+\int_{\Omega_\varrho}|x|^{-\alpha}u^{q(x)}\psi_r^\gamma(x)\ln\frac{u}{m(\varrho)}dx\\&\le\mu_2\gamma\int_{\Omega_\varrho}|\nabla u|^{p(x)-1}|\nabla\psi_r|\psi_r^{\gamma-1}(x)\ln\frac{u}{m(\varrho)}dx.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}&\mu_2\gamma\int_{\Omega_\varrho}|\nabla u|^{p(x)-1}|\nabla\psi_r|\psi_r^{\gamma-1}(x)\ln\frac{u}{m(\varrho)}dx\\&\le C(\varepsilon_4)\int_{\Omega_\varrho}u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\left(\ln\frac{u}{m(\varrho)}\right)^{p(x)}dx+\mu_2\gamma\varepsilon_4\int_{\Omega_\varrho}\psi_r^\gamma u^{-1}|\nabla u|^{p(x)}dx,\end{aligned}$$

take $\varepsilon_4=\frac{\mu_1}{2\mu_2\gamma}$, then

$$\begin{aligned}&\frac{\mu_1}{2}\int_{\Omega_\varrho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx+\int_{\Omega_\varrho}|x|^{-\alpha}u^{q(x)}\psi_r^\gamma(x)\ln\frac{u}{m(\rho)}dx\\&\le C(\mu_1,\mu_2,\gamma)\int_{\Omega_\varrho}u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\left(\ln\frac{u}{m(\rho)}\right)^{p(x)}dx.\end{aligned}$$

Further,

$$\begin{aligned}&\int_{\Omega_\varrho}u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\left(\ln\frac{u}{m(\rho)}\right)^{p(x)}dx\\&\le C(\varepsilon_5)\int_{\Omega_\varrho}|x|^{\frac{\alpha q(x)}{q(x)-p(x)+1}-\alpha}\left(\ln\frac{u}{m(\varrho)}\right)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}|\nabla\psi_r|^{\frac{p(x)q(x)}{q(x)-p(x)+1}}dx\\&\quad+\varepsilon_5\int_{\Omega_\varrho}|x|^{-\alpha}\ln\frac{u}{m(\varrho)}u^{q(x)}\psi_r^{\frac{(\gamma-p(x))q(x)}{p(x)-1}}dx.\end{aligned}$$

Take $\varepsilon_5=\frac{1}{2C(\mu_1,\mu_2,\gamma)}$. Since $\frac{(\gamma-p(x))q(x)}{p(x)-1}>\gamma$, $\psi_r(x)\le 1$, we have

$$\begin{aligned}&\frac{\mu_1}{2}\int_{\Omega_\varrho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx+\frac{1}{2}\int_{\Omega_\varrho}|x|^{-\alpha}u^{q(x)}\psi_r^\gamma(x)\ln\frac{u}{m(\varrho)}dx\\&\le C(\mu_1,\mu_2,\gamma)\int_{\Omega_\varrho\cap\{x:r\le|x|\le\sqrt{r}\}}|x|^{\frac{\alpha q(x)}{q(x)-p(x)+1}-\alpha}\left(\ln\frac{u}{m(\varrho)}\right)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}|\nabla\psi_r|^{\frac{p(x)q(x)}{q(x)-p(x)+1}}dx.\end{aligned}\tag{3.13}$$

By Lemma 2.1, we get $0 < 1 + \frac{(p_R^+ - 1)q_R^+}{q_R^- - p_R^+ + 1} < \infty$. Denote $\lambda = \sup_{x\in\Omega}\left(\frac{(p(x)-\alpha)q(x)}{q(x)-p(x)+1} + \alpha\right)$, and from Theorem 3.1 and (3.13), we have

$$
\begin{aligned}
&\frac{\mu_1}{2}\int_{\Omega_\varrho} \frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx + \frac{1}{2}\int_{\Omega_\varrho} |x|^{-\alpha}u^{q(x)}\psi_r^\gamma(x)\ln\frac{u}{m(\varrho)}dx \\
&\leq C\int_{\Omega_\varrho\cap\{x:r\leq|x|\leq\sqrt{r}\}} |x|^{\frac{\alpha q(x)}{q(x)-p(x)+1}-\alpha}\left(\ln|x|^{-Q}+C\right)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}\left(\frac{2}{|x|\ln\frac{1}{r}}\right)^{\frac{p(x)q(x)}{q(x)-p(x)+1}}dx \\
&\leq C\left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\int_{\Omega_\varrho\cap\{x:r\leq|x|\leq\sqrt{r}\}} |x|^{\frac{\alpha q(x)}{q(x)-p(x)+1}-\alpha}\left[\left(\ln\frac{1}{|x|}\right)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}+1\right]\left(\frac{1}{|x|}\right)^{\frac{p(x)q(x)}{q(x)-p(x)+1}}dx \\
&\leq C\left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\int_{\Omega_\rho\cap\{x:r\leq|x|\leq\sqrt{r}\}}\left(\ln\frac{1}{|x|}\right)^{1+\frac{(p_R^+-1)q_R^+}{q_R^--p_R^++1}}\left(\frac{1}{|x|}\right)^{\lambda}dx \\
&\leq C\left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\int_r^{\sqrt{r}}\left(\frac{1}{t}\right)^{\lambda}\left(\ln\frac{1}{t}\right)^{1+\frac{(p_R^+-1)q_R^+}{q_R^--p_R^++1}}t^{N-1}dt,
\end{aligned}
$$

where $C = C\left(N, \mu_1, \mu_2, \gamma, p_R^+, p_R^-, q_R^-, q_R^+, R\right)$.

Further, by (1.6), we get $\lambda < N$, then

$$
\begin{aligned}
&\left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\int_r^{\sqrt{r}}\left(\frac{1}{t}\right)^{\lambda}\left(\ln\frac{1}{t}\right)^{1+\frac{\left(p_R^+-1\right)q_R^+}{q_R^--p_R^++1}}t^{N-1}dt \\
&\leq \left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\left(\ln\frac{1}{r}\right)^{1+\frac{\left(p_R^+-1\right)q_R^+}{q_R^--p_R^++1}}\int_r^{\sqrt{r}} t^{N-1-\lambda}dt \\
&= \left(\ln\frac{1}{r}\right)^{-\frac{q_R^-p_R^-}{q_R^+-p_R^-+1}}\left(\ln\frac{1}{r}\right)^{1+\frac{\left(p_R^+-1\right)q_R^+}{q_R^--p_R^++1}}\frac{1}{N-\lambda}r^{\frac{1}{2}(N-\lambda)}\left(1-r^{\frac{1}{2}(N-\lambda)}\right) \\
&\to 0,
\end{aligned}
$$

as $r \to 0$. Therefore, we obtain

$$
\lim_{r\to 0}\frac{\mu_1}{2}\int_{\Omega_\varrho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx + \frac{1}{2}\int_{\Omega_\varrho}|x|^{-\alpha}\,u^{q(x)}\psi_r^\gamma(x)\ln\frac{u}{m(\varrho)}dx \leq 0,
$$

then

$$
\mu_1\int_{\Omega_\varrho}\frac{|\nabla u|^{p(x)}}{u}dx + \int_{\Omega_\varrho}|x|^{-\alpha}u^{q(x)}\ln\frac{u}{m(\varrho)}dx = 0.
$$

Hence $u(x) = m(\varrho)$ almost everywhere in Ω_ϱ and the Lebesgue measure of Ω_ϱ equals to zero. Considering further the function $-u(x)$ instead of $u(x)$, we obtain the boundedness of $-u(x)$ in a neighborhood of the point 0. Thus we have proved that $u \in L^\infty(\Omega)$.

Next, we take the test function

$$\widetilde{\varphi} = \psi^{p^+} u,$$

where $\psi \equiv 1$ in $B_{2\rho}(0) \backslash B_\rho(0)$, $\psi \equiv 0$ outside $B_{\frac{5\rho}{2}}(0) \backslash B_{\frac{\rho}{2}}(0)$, $0 \leq \psi(x) \leq 1$, $|\nabla\psi| \leq \frac{C}{\rho}$ and $0 < \rho \leq 1$. Testing the equality (2.3) by $\widetilde{\varphi}$, we have

$$\int_\Omega A(x,u,\nabla u)\left(p^+\psi^{p^+-1}u\nabla\psi + \psi^{p^+}\nabla u\right) + g(x,u)\psi^{p^+}udx = 0.$$

By virtue of the conditions $(1.2)-(1.5)$, we have

$$\begin{aligned}
&\int_{B_{\frac{5\rho}{2}}(0)} \mu_1|\nabla u|^{p(x)}\psi^{p^+} + |x|^{-\alpha}|u|^{q(x)+1}\psi^{p^+}dx \\
&\leq p^+\mu_2\int_{B_{\frac{5\rho}{2}}(0)} |\nabla u|^{p(x)-1}\psi^{p^+-1}|\nabla\psi||u|dx \\
&= p^+\mu_2\int_{B_{\frac{5\rho}{2}}(0)} \left[|\nabla\psi||u|\psi^{p^+-1-\frac{p^+}{p'(x)}}\right]\left[|\nabla u|^{p(x)-1}\psi^{\frac{p^+}{p'(x)}}\right]dx \\
&\leq C(\mu_2,p^+,\varepsilon_6)\int_{B_{\frac{5\rho}{2}}(0)} |\nabla\psi|^{p(x)}|u|^{p(x)}\psi^{p^+-p(x)}dx + p^+\mu_2\varepsilon_6\int_{B_{\frac{5\rho}{2}}(0)} |\nabla u|^{p(x)}\psi^{p^+}dx.
\end{aligned}$$

Take $\varepsilon_6 = \frac{\mu_1}{2p^+\mu_2}$, we have

$$\begin{aligned}
\int_{B_{\frac{5\rho}{2}}(0)} |\nabla u|^{p(x)}\psi^{p^+}dx &\leq C(\mu_1,\mu_2,p^+)\int_{B_{\frac{5\rho}{2}}(0)} |\nabla\psi|^{p(x)}|u|^{p(x)}\psi^{p^+-p(x)}dx \\
&\leq C\frac{1}{\rho^{p^+}}\max\left\{||u||_\infty^{p^+}, ||u||_\infty^{p^-}\right\}\left|B_{\frac{5\rho}{2}}(0)\right| \\
&\leq C\frac{1}{\rho^{p^+}}\omega_n\left(\frac{5\rho}{2}\right)^N \\
&= C(\mu_1,\mu_2,p^+)\rho^{N-p^+},
\end{aligned}$$

where ω_n is the volume of the unit ball, $|B_{\frac{5\rho}{2}}(0)|$ is the volume of the ball $B_{\frac{5\rho}{2}}(0)$.

Further,

$$\int_{B_{2\rho}(0)\backslash B_\rho(0)} |\nabla u|^{p(x)}dx \leq C(\mu_1,\mu_2,p^+)\rho^{N-p^+}, \tag{3.14}$$

then we obtain

$$\begin{aligned}
\int_{B_\rho(0)} |\nabla u|^{p(x)}dx &= \sum_{j=1}^{\infty} \int_{B_{2^{1-j}\rho}(0)\setminus B_{2^{-j}\rho}(0)} |\nabla u|^{p(x)}dx \\
&\le C\sum_{j=1}^{\infty} \left(2^{-j}\rho\right)^{N-p^+} \\
&\le C(\mu_1,\mu_2,p^+)\rho^{N-p^+} \\
&\to 0,
\end{aligned}$$

as $\rho \to 0$. So $|\nabla u| \in L^{p(x)}(\Omega)$.

Thus, we have proved that $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Next, we will show that $u(x)$ is a solution of equation (1.1) in the domain Ω. Pick $\eta_\rho \in C_0^\infty(R^N)$ be the cutoff function for the ball $B_\rho(0)$, $\eta_\rho \equiv 1$ in $B_\rho(0)$, $\eta_\rho \equiv 0$ outside the ball $B_{2\rho}(0)$, $|\nabla \eta_\rho| \le \frac{C}{\rho}$ and $0 < \rho \le 1$. Let $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Testing the equation (2.3) by the test function $(1-\eta_\rho)\varphi$, we have

$$\int_\Omega A(x,u,\nabla u)\nabla[(1-\eta_\rho)\varphi]dx + \int_\Omega g(x,u)(1-\eta_\rho)\varphi dx = 0,$$

that is,

$$\int_\Omega A(x,u,\nabla u)(1-\eta_\rho)\nabla\varphi dx - \int_\Omega A(x,u,\nabla u)\varphi\nabla\eta_\rho dx + \int_\Omega g(x,u)(1-\eta_\rho)\varphi dx = 0.$$

Indeed,

$$\begin{aligned}
|A(x,u,\nabla u)(1-\eta_\rho)\nabla\varphi| &\le \mu_2|\nabla u|^{p(x)-1}|\nabla\varphi| \\
&\le \mu_2\left(\frac{p(x)-1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|\nabla\varphi|^{p(x)}\right) \\
&\in L^1(\Omega),
\end{aligned}$$

therefore, by Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{\rho\to 0}\int_\Omega A(x,u,\nabla u)(1-\eta_\rho)\nabla\varphi dx = \int_\Omega A(x,u,\nabla u)\nabla\varphi dx.$$

In the same way,

$$\lim_{\rho\to 0}\int_\Omega g(x,u)(1-\eta_\rho)\varphi dx = \int_\Omega g(x,u)\varphi dx.$$

Meanwhile, by (3.14), we have

$$
\begin{aligned}
&\left|\int_\Omega A(x,u,\nabla u)\varphi\nabla\eta_\rho dx\right| \\
&\le \frac{C\mu_2}{\rho}\int_{B_{2\rho}(0)\setminus B_\rho(0)}|\nabla u|^{p(x)-1}dx \\
&\le \frac{C(\mu_2)}{\rho}\parallel |\nabla u|^{p(x)-1}\parallel_{L^{\frac{p(x)}{p(x)-1}}(B_{2\rho}(0)\setminus B_\rho(0))}\parallel 1\parallel_{L^{p(x)}(B_{2\rho}(0)\setminus B_\rho(0))} \\
&\le \frac{C(\mu_2)}{\rho}\left[\int_{B_{2\rho}(0)\setminus B_\rho(0)}|\nabla u|^{p(x)}dx\right]^{\frac{p^--1}{p^+}}\cdot|B_{2\rho}(0)\setminus B_\rho(0)|^{\frac{1}{p^+}} \\
&\le \frac{C(\mu_1,\mu_2,p^+)}{\rho}\rho^{\frac{\left(p^--1\right)\left(N-p^+\right)}{p^+}}\left(\rho^N\right)^{\frac{1}{p^+}} \\
&= C(\mu_1,\mu_2,p^+)\rho^{\frac{p^-\left(N-p^+\right)}{p^+}} \\
&\to 0,
\end{aligned}
$$

as $\rho\to 0$.

So we have obtained that equality (2.3) is fulfilled for any test function.

Therefore, the isolated singular point 0 is removable for solutions of equation (1.1).

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(Received July 5, 2013)